

Defining Electromagnetic Fields from Continuity and Divergence-Free Structure

Two snapshots of a single scalar field define flow

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One-Sentence Summary. Continuity and divergence-free structure force electromagnetic energy to be described by directional, circulating degrees of freedom; however, energy density and flux alone do not uniquely determine the electric and magnetic fields without additional relational data.

Abstract. We clarify the conceptual bridge between a field-only ontology of electromagnetic energy flow and the usual electric and magnetic field variables. We show rigorously that a single scalar energy density does not define propagation, but that the relation between scalar configurations at different times, constrained by continuity, forces the introduction of directional transport. We prove a reconstruction result: given an energy density u and an energy flux S satisfying $|S| \leq c u$, one can always construct at least one pair of fields E and B that reproduce u and S . The non-uniqueness of this reconstruction matches the true physical degrees of freedom identified as polarization. We conclude with a precise statement of what is implied by continuity and what requires the full Maxwell dynamical law.

Keywords. Maxwell theory, continuity equation, divergence-free fields, Poynting vector, energy density, field reconstruction, circulation, topology

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1. Motivation

A recurring conceptual objection is:

If electromagnetic energy is taken as primary, and energy density is a scalar field, how do vector fields such as \mathbf{E} and \mathbf{B} arise?

The objection is valid if one assumes that vectors must be extracted directly from a single scalar configuration. They cannot.

A single scalar field does not define flow. Flow is defined by the **relation between scalar configurations at different times under continuity**.

This distinction is central and non-negotiable.

The correct ordering is:

- a scalar energy density $u(\mathbf{x}, t)$ is an observable of a process,
- continuity relates changes of that observable across time,
- the relation between two scalar configurations requires directional transport,
- directional transport is necessarily described in vectorial language.

Thus, vectors do not arise from scalars by inspection. They arise from **change under constraint**.

This document makes that chain explicit and states precisely what can be reconstructed from energy density and energy flow alone.

2. Assumptions

We assume a source-free region.

We assume that electromagnetic energy admits:

1. A nonnegative energy density $u(\mathbf{x}, t) \geq 0$.
2. An energy flux $\mathbf{S}(\mathbf{x}, t)$ satisfying a continuity description

$$\partial_t u + \nabla \cdot \mathbf{S} = 0.$$

This equation expresses the observed fact that when energy moves through space, it does so continuously: changes in density are accounted for by transport.

We also assume the empirically observed causal bound

$$|\mathbf{S}| \leq c u,$$

which states that the rate of energy transport does not exceed the maximal observed propagation rate. This is an operational fact about how energy is seen to move through space, not a restriction on the possible ways transport may be organized or described.

We do not assume particles, matter, constitutive media, quantum axioms, or mechanical primitives.

We do not derive Maxwell's dynamical equations from continuity alone. We instead establish what continuity and divergence-free structure *force*, and what additional structure is required to describe evolution.

3. Why a scalar field alone cannot encode propagation

A scalar field $u(\mathbf{x}, t)$ at a single time slice contains no information about direction. It specifies magnitude only.

Two scalar snapshots, however, allow us to define transport. Given $u(\mathbf{x}, t_1)$ and $u(\mathbf{x}, t_2)$, continuity requires that differences between them be explained by motion through space.

That motion cannot be described without direction.

Thus, the continuity equation does not close on u alone. It introduces \mathbf{S} as a necessary relational quantity. The pair (u, \mathbf{S}) is the minimal description of conserved flow.

Vectors are therefore not additional structure. They are the minimal language required to describe change consistently.

4. Divergence-free structure and circulation in 3D

In three dimensions, divergence-free vector fields have a canonical geometric feature: they admit circulation and vortex-like organization.

The mathematical statement is standard:

If a vector field \mathbf{F} is divergence-free on a simply connected region, then there exists a vector potential \mathbf{A} such that

$$\mathbf{F} = \nabla \times \mathbf{A}.$$

This is a general theorem about solenoidal fields. Locally, divergence-free structure is equivalent to curl structure and hence to circulation. Globally, the existence of nontrivial circulation depends on the topology of the domain.

This distinction underlies the appearance of toroidal shells and winding numbers: in three dimensions, divergence-free structure naturally organizes into tubes and closed surfaces when the topology permits it.

5. What it means to define the electromagnetic field

There are two distinct tasks that must be separated:

1. Representation: showing that (u, \mathbf{S}) can be represented by fields \mathbf{E} and \mathbf{B} .
2. Dynamics: specifying how \mathbf{E} and \mathbf{B} evolve in time.

Continuity addresses the first task partially and constrains the second, but does not uniquely determine dynamics. This document focuses exclusively on the representation problem.

6. Reconstruction lemma: from (u, \mathbf{S}) to (\mathbf{E}, \mathbf{B})

We now show that if (u, \mathbf{S}) obeys $|\mathbf{S}| \leq cu$, then one can always construct at least one pair of fields \mathbf{E} and \mathbf{B} satisfying the standard electromagnetic energy and flux relations.

6.1. Lemma (existence, non-uniqueness)

Given scalar $u > 0$ and vector \mathbf{S} with $|\mathbf{S}| \leq cu$, there exist vectors \mathbf{E} and \mathbf{B} such that

$$u = \frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{1}{2\mu_0} |\mathbf{B}|^2,$$

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}.$$

The pair (\mathbf{E}, \mathbf{B}) is not unique. There is an infinite family of solutions corresponding to polarization and duality degrees of freedom.

6.2. Construction

Let $\hat{\mathbf{s}}$ be a unit vector in the direction of \mathbf{S} if $\mathbf{S} \neq 0$; if $\mathbf{S} = 0$, choose any unit vector.

Choose a unit vector $\hat{\mathbf{e}}$ orthogonal to $\hat{\mathbf{s}}$ and define $\hat{\mathbf{b}} := \hat{\mathbf{s}} \times \hat{\mathbf{e}}$, forming a right-handed orthonormal frame.

Define

$$\mathbf{E} := E \hat{\mathbf{e}}, \quad \mathbf{B} := B \hat{\mathbf{b}}.$$

Then

$$\mathbf{E} \times \mathbf{B} = EB \hat{\mathbf{s}}.$$

Imposing the flux condition yields

$$\frac{1}{\mu_0} EB = |\mathbf{S}|.$$

The energy condition gives

$$u = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2.$$

Using $c^2 = 1/(\mu_0\epsilon_0)$ and eliminating B leads to a quadratic equation for E^2 with real solutions whenever $|\mathbf{S}| \leq cu$. Choosing either root completes the construction.

6.3. Interpretation

- The inequality $|\mathbf{S}| \leq cu$ is exactly the condition required for reconstruction.
- The choice of transverse frame corresponds to polarization freedom.
- Continuous duality rotations leave (u, \mathbf{S}) invariant.

Thus, (u, \mathbf{S}) underdetermines (\mathbf{E}, \mathbf{B}) by exactly two local degrees of freedom. This matches the two missing local degrees of freedom identified as polarization.

7. Why orthogonal coupling appears

The frequent question “why does one generate the other orthogonally?” should be recast as:

Why does divergence-free propagation require rotational coupling?

The minimal structural answer is:

- To propagate while remaining divergence-free, degrees of freedom must rotate into one another. In three dimensions, this rotation is encoded by curl.
- Maxwell evolution uses curl operators, which are sufficient to transport divergence-free structure without creating sources or sinks.

Continuity alone does not fix this evolution uniquely; curl dynamics is a minimal realization of divergence-free propagation.

In this sense, electric and magnetic fields are names for the two interlocked rotational aspects of a single propagating, divergence-free energy flow.

8. Where toroidal winding numbers (m, n) enter

The reconstruction above is local. Toroidal winding numbers are global.

On an invariant torus T^2 , a smooth source-free tangent flow decomposes into windings around the two fundamental cycles. For example, a linear flow of rational slope on T^2 closes after (m, n) windings, enforcing discrete circulation classes.

The integers (m, n) do not define the electric or magnetic fields pointwise.

They characterize the global organization of a particular energy flow: how that flow wraps, closes, and circulates on a toroidal surface.

The fields \mathbf{E} and \mathbf{B} are a different kind of description. They provide a local, differential account of how energy is transported and rotated through space.

These are not competing structures. They are complementary descriptions of the same underlying process.

In the same way that energy and frequency describe the same electromagnetic phenomenon from different perspectives, (m, n) describe the global pattern of a flow, while (\mathbf{E}, \mathbf{B}) describe its local transport.

9. Conclusions

- A scalar energy density alone cannot encode propagation.
- Continuity forces directional transport and hence vector structure.

- Divergence-free structure locally forces circulation in three dimensions.
- Given (u, \mathbf{S}) with $|\mathbf{S}| \leq cu$, one can construct \mathbf{E} and \mathbf{B} reproducing the same energy and flux.
- The reconstruction is necessarily non-unique, matching polarization degrees of freedom.

10. Closing statement

A scalar energy observable does not generate electromagnetic vector fields by itself. The evolution of energy under continuity forces transport, and transport forces directional, circulating structure in three dimensions.

Electric and magnetic fields provide a minimal local representation of this structure while reproducing the observable pair (u, \mathbf{S}) .

This provides a precise conceptual bridge between a flow-first Maxwell ontology and the standard (\mathbf{E}, \mathbf{B}) description.